The ZIENA Solver for American Options Pricing

Jorge Nocedal (February 2008)

This paper considers the use of the ZIENA software engine for American options pricing. Its robustness, speed, and ease of use make it a powerful tool for real-time trading of options. The ZIENA solver can be used for classical Black-Scholes-Merton models as well as models with stochastic volatility. It makes use of the *partial differential variational inequality* formulation of the problem.

Most options traded on options exchanges world-wide and a large fraction of options traded over-the-counter are American-style, including options on stocks of individual companies, stock indexes, foreign currencies, interest rates, commodities, and energy. Options books of a large financial institution may contain options on thousands of different underlying assets, perhaps several dozen different contracts with different contract terms (such as different expiration dates, ranging from days to many years, and different strike prices) for each underlying asset, with a total of possibly tens of thousands of positions. As the underlying asset prices change in real time throughout the trading day, the options prices change as well. Re-pricing a large options book may require re-computing thousands of options prices in real time. For such large scale applications, CPU times required to price a single option contract have to be a fraction of a second. Here is where the ZIENA solver shows its strength: it is much faster and accurate than the methods used at present.

Formulation. Consider an American put option with strike price K and maturity T. If the option is exercised when the underlying asset price is S, the option holder receives

$$\Psi(S) = (K - S)^+ = \max(K - S, 0).$$

Similarly, the payoff function for an American call option is

$$\Psi(S) = (S - K)^+.$$

Let $V(t, S)$ be the option value at time t when the asset price is S. Then V solves the following *partial differential variational inequality (PDVI)*:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV \leq 0, \quad t \in [0, T), S \in (0, \infty),$$

$$V \geq \Psi, \quad t \in [0, T), S \in (0, \infty),$$

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV) \cdot (V - \psi) = 0, \quad t \in [0, T), S \in (0, \infty)$$

subject to terminal condition

$$V(T, S) = \Psi(S), \quad S \in (0, \infty),$$

where r is the risk free interest rate, q is the dividend yield paid by the underlying asset.

After these equations are discretized (both in space and time) one obtains a so-called *linear complementarity problem*. The ZIENA solver is applied to this problem. To illustrate its speed and scalability, a model was developed with different choices for volatility, maturity and for different number of time steps in the discretization. The ZIENA solver is compared with the Projected-SOR method (P-SOR), which is the method of choice today in options exchanges.

Volatility	Maturity	Time Steps	P-SOR	Ziena
0.4	0.5	640	28 sec	1.7 sec
0.4	5.0	640	90 sec	2.1 sec
0.2	0.5	320	3.5 sec	0.5 sec
0.2	5.0	640	93 sec	2.1 sec

As these results show, the ZIENA solver is much faster than the projected SOR method in all scenarios. The tests reported above imposed very high accuracy (10^{-9}) in the solution; the CPU times decrease by a factor of 5 if only 1 cent of accurace is required.

The ZIENA solver is a premier software package for pricing American options. It is available as a thread-safe, embeddable software library on multiple platforms, with programmatic APIs and interfaces to major modeling languages. Full support and continued development of the ZIENA options pricing engine is provided by Ziena Optimization, Inc. For more information, visit *http://www.ziena.com*.